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The end-effect error in the determination of thermal conductivity using a hot-wire apparatus

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Abstract—Most thermal conductivity measuring instruments based on the transient line source technique make provision for compensating for the end-effect in a practical manner. A theoretical analysis shows that the error caused by the end-effect can be kept very small, so that it is not always necessary to correct for this in a practical manner. An upper limit for the error is derived. In the analysis a distinction is made between the conductive and the convective end-effects; in the first part of the measurement the conductive effect dominates, but its influence is rapidly overtaken by that of the convective effect.

1. INTRODUCTION

THERMAL conductivity measurements are frequently done with a hot-wire apparatus. Such an apparatus consists of a thin platinum wire immersed in a liquid whose thermal conductivity has to be determined. An electrical current is used to generate heat in the wire. The resulting temperature rise of this wire is inversely proportional to the thermal conductivity of the liquid, and will change the resistance of this wire slightly. This resistance change is measured using a Wheatstone or other resistance bridge. When the bridge output voltage is plotted against a logarithmic time-base, a straight line is found whose slope is determined by the thermal conductivity of the liquid. In this procedure it is assumed that the wire has a uniform temperature, but this is not correct because its ends are being cooled down by the wire-supports. The measured temperature rise is thus smaller than expected and so the real thermal conductivity is smaller than the measured value.

It is common practice to compensate for this effect in a practical manner by either one of the following methods:

—*The compensating lead method* [1–8] employs two wires in different branches of a Wheatstone bridge so that a subtracting effect deletes the end-effects and the response of the bridge is that of a part of an infinite wire. A great disadvantage of this method is that the wires must be matched with respect to their resistance-to-length ratios to assure the same heat production per unit length in long and short wires. Kestin and Wakeham [4] report that for wires of 0.15 cm in length taken from the same roll, these ratios may differ by as much as 4%! This compensating method can be used to compensate simultaneously for thermocouple and strain effects, as discussed in [6].

—*The potential lead method* [9–14] employs two potential taps at c. 20 mm from the ends of the wire. These taps consist of a wire even thinner than the hot-wire to which they are spot-welded. The temperature rise of the hot-wire is said to be very little influenced by this connection. This method is inconvenient to use when taking measurements of electrically conducting liquids [8] because of insulation problems.

In the papers referred to, such practical corrections are applied because an analysis that takes the difficult experimental boundary conditions into account is not available. The relative method, in which two Wheatstone bridges are used, also partially eliminates the end-effect [15].

In previous studies of the end-effect [16–20] only conduction was taken into account. In this paper it is shown that a convective end-effect constitutes the main error in those cases in which a measurement is continued 'until the onset of convection'; only for very short measuring times is the convective end-effect negligible. The conductive end-effect was first studied by Blackwell [16, 17], who considered the temperature distribution outside a circular cylinder in the following cases:

(i) Transient heat flow in the infinite region bounded internally by an (infinite) circular cylinder. There is a constant flux of heat across a finite length of the internal boundary surface and the remainder of this surface is insulated.
(ii) Transient heat flow in the semi-infinite region bounded internally by a circular cylinder and by planes at right angles to the axis of the cylinder. There is a constant flux of heat across the cylindrical boundary and the planes are maintained at zero temperature.
(iii) As for case (i), with constant heat production in the cylinder.

Case (i) would give the most appropriate error for a potential lead correction where the potential leads have the same thickness as the wire, since in both cases the temperature rise at the end is half that of the infinite part of the wire.

NOMENCLATURE

a	radius of the hot-wire
A	$g\beta QPr/2\pi a\lambda\nu(Pr-1)$
C	$e^{0.5772}$ (0.5772 = Euler's constant)
C_1	integration constant
C_2	integration constant
erf	error function
E	error due to the end-effect
g	gravitational constant
i	$\sqrt{-1}$
ln	natural logarithm
K_0	Bessel function
K_1	Bessel function
l_c	effective length of the conductive end-effect
l_p	penetration length
L	length of the wire
$\mathscr{L}$	Laplace operator
p	Laplace domain variable
Pr	Prandtl number
q	heat sink due to conduction to the environment
Q	heat production in the wire per unit length
r	radial distance from the center of the wire
R	r/a
s	penetration length in the Laplace domain
t	variable for time
t_e	time after which measurement is stopped
T	temperature variable
u	velocity of the liquid around the wire
v	Laplace transform of u
V	$a\sqrt{x/\kappa}$
x	radial variable in complex plane
z	variable for the distance from the end of the wire.

Greek symbols

α	heat loss coefficient
β	coefficient for cubical expansion, $\alpha\kappa_1/l(\pi\alpha^2\lambda_1)$
∂	operator for partial differentiation
ξ	radius in complex plane
θ	Laplace transform of the temperature
κ	thermal diffusivity
κ_1	thermal diffusivity of the wire
λ	thermal conductivity
λ_1	thermal conductivity of the wire
μ	dynamic viscosity
ν	kinematic viscosity
π	3.141592
ρ	density
τ	$\kappa t/a^2$
ϕ_1	axial heat flux in the liquid
ϕ_ω	axial heat flux in the wire
∞	infinity.

Case (ii), with the assumption made in (iii), would be a 'worst case' analysis for a wire soldered to its supports.

The much-referred-to analyses of Healy *et al.* [18] and Haarman [19] are both based on the work of Blackwell. Haarman gave a rough estimation of the conductive end-effect using a steady-state temperature distribution. His result is the same as the conductive end-effect determined in the time-dependent analysis in Section 3 of this paper.

Horrocks and McLaughlin [20] also used a steady-state temperature distribution in their estimation, but ended up with a formula containing the outer cell dimensions, which is not very appropriate.

A decrease in density of the liquid in the thermal boundary layer around the wire is the driving force for a convectional current which moves this thermal boundary layer upwards. This results in an inflow of cold liquid at the bottom, causing the bottom end-effect to increase. The same mechanism will cause the top end-effect to decrease. Initially the increase and decrease will roughly cancel each other out as far as the average temperature of the wire is concerned. At a certain stage, however, the convective current will have penetrated so far along the wire, that near the top of the wire the end-effect will have vanished almost completely while at the bottom it will still be growing. A quantitative analysis of this convective end-effect is given in Section 4. It is remarkable that this analysis has never been done before; most thermal conductivity measurements done in the past are likely to have been plagued by this type of end-effect rather than by the conductive end-effect. Only since the introduction of fast electronics have measurement times become so short that convection now plays a minor role.

The 'onset of convection' is a phenomenon that is still not well understood. In the last section a possible cause of this effect is given, and other consequences of convective motion are discussed.

2. DEFINITION OF THE END-EFFECT

An electrical current sent through an infinitely long, thin wire immersed in a liquid will cause a temperature rise, the extent of which depends on the properties of the wire and the liquid [18]. A finite wire, however, will be cooled down by its supports, thus causing a temperature defect near its ends (see Fig. 1).

The measured average temperature rise of the wire is therefore slightly smaller than that predicted by the theory. This difference causes the end-effect. The relative end-effect error is therefore defined as the difference in surface area between the two temperature

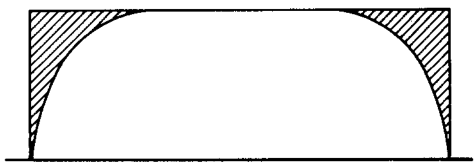

FIG. 1. Theoretical and practical temperature distributions over the length of the wire. The hatched areas represent the end-effect error.

distributions in Fig. 1, relative to the absolute area under the theoretical curve. The size of the relative end-effect error will be studied in the sections that follow.

The average temperature rise of the wire is found by measuring the change in the wire's resistance, which is directly proportional to it. In practice this means that the change in resistance is subject to the same error.

3. ERROR DUE TO THE CONDUCTIVE END-EFFECT

A worst-case analysis for a wire soldered to 'cold' supports is given in this section. The problem is described as follows:

—Transient heat flow in the semi-infinite region bounded internally by a circular cylinder and by planes at right angles to the axis of the cylinder. There is a constant rate of heat production in the cylinder.

Axial conduction in the liquid is assumed to be negligible. In the Appendix an analysis is given in which this last assumption is discussed.

Note that even if the axial heat flow is not negligible, taking this effect into account is misleading since convectional currents are very likely to disturb the temperature profile in the liquid. This statement of the problem, which is less complicated than those proposed by Blackwell, is therefore allowed without compromising the accuracy at all.

To determine the end-effect, an expression for the temperature rise of the wire as a function of the distance from the end of the wire must be found when a certain amount of heat per unit length is produced in the wire. The equation governing the temperature of the wire near its end is:

$$\frac{1}{\kappa_1}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2} + \frac{Q-q}{\pi a^2 \lambda_1}, \quad T = T(t,z) \tag{1}$$

with:

$$T(0,z) = 0 \tag{2}$$

$$T(t,0) = 0 \tag{3}$$

$$\lim_{z\to\infty} \frac{\partial T(t,z)}{\partial z} = 0. \tag{4}$$

Condition (3) states that the actual wire support has an infinite heat capacity (compared to the wire). This is usually the case when the end of the wire is soldered to its supports. When potential leads of the same thickness are spot-welded (with no addition of material) to the wire at a distance sufficiently far from the end, this condition should read $T(t,0) = T(t,\infty)/2$; in agreement with Haarman [19], we can then readily conclude that the end-effect is half of that of the case we want to solve. Normally supports have a thermal mass in between these two cases. The error resulting from this analysis can be reduced by a certain percentage depending on the heat capacity of the support. The term q represents heat loss caused by conduction to the surrounding liquid

$$q = \alpha T. \tag{5}$$

The heat loss coefficient α, which is thus a constant in the solution, actually varies slowly from its initial value to its end value, but for this purpose it can be assumed to be constant. It is obvious, however, that the smallest value for α will give the upper limit for the end-effect, and the biggest value the lower limit. How such a value can be found will be shown at the end of this section.

Calculation of the conductive end-effect is expected to be done in this manner:

—calculation of the temperature distribution in the wire as a function of time and place, followed by
—integration over the temperature defect $[T(t,\infty) - T(t,z)]$ from $z = 0$ to $z = \infty$ and division by $L \cdot T(t,\infty)$.

The first step is analytically not possible, and this is likely to be the reason why this problem could not be solved previously. If, however, the following line of calculation is followed, the result is obtained by standard mathematics:

—calculation of the temperature distribution in the wire as a function of the Laplace variable and place, followed by
—integration in the Laplace domain over the temperature defect from $z(p,0)$ to $z(p,\infty)$, followed by division by $L \cdot T(t,\infty)$, and then
—transformation to the time domain.

The calculation now given follows from this scheme.

After Laplace transform of equation (1) an ordinary differential equation is obtained

$$\frac{d^2\theta}{dz^2} - \left(\frac{p}{\kappa_1} + \frac{\alpha}{\pi a^2 \lambda_1}\right)\theta = -\frac{Q}{\pi a^2 \lambda_1 p} \tag{6}$$

whose solution can easily be found, using the transforms of the boundary conditions (3) and (4)

$$\theta(p,z) = \frac{Q\kappa_1}{\pi a^2 \lambda_1} \times \left\{\frac{1-\exp\left[-(z/\sqrt{\kappa_1})\sqrt{p+(\alpha\kappa_1/\pi a^2\lambda_1)}\right]}{p(p+\alpha\kappa_1/\pi a^2\lambda_1)}\right\}. \tag{7}$$

Equation (7) describes the temperature profile in the Laplace domain. From the definition in Section 2, we find that the end-effect is obtained by calculating

$$E = \frac{2}{L \cdot T(t,\infty)} \int_0^\infty [T(t,\infty) - T(t,z)]\,dz \tag{8}$$

where the factor 2 shows that both ends of the wire are taken into account.

Since determining the actual temperature profile is not the basic aim of this study, the inverse Laplace transform followed by the integration will now be interchanged using

$$\int_0^\infty \mathscr{L}^{-1}\{f(p,z)\}\,dz = \mathscr{L}^{-1}\left\{\int_0^\infty f(p,z)\,dz\right\}. \quad (9)$$

The error due to the end-effect is then described by

$$E = \frac{2\mathscr{L}^{-1}\int_0^\infty \{\exp[(z/\sqrt{\kappa_1})\sqrt{p+\beta}]/p(p+\beta)\,dz\}}{L\cdot\mathscr{L}^{-1}\{1/p(p+\beta)\}}$$

$$= \frac{2\mathscr{L}^{-1}\{\sqrt{\kappa_1}/p(p+\beta)^{3/2}\}}{L\cdot\mathscr{L}^{-1}\{1/p(p+\beta)\}} \quad (10)$$

where

$$\beta = \frac{\alpha\kappa_1}{\pi a^2\lambda_1}. \quad (11)$$

After the inverse transforms have been carried out, this reduces to

$$E = \frac{2}{L}\sqrt{\kappa_1/\beta}\left(\frac{\operatorname{erf}\sqrt{\beta t} - 2\sqrt{\beta t/\pi}\,e^{-\beta t}}{1-e^{-\beta t}}\right). \quad (12)$$

A minimum value for the heat loss coefficient α still has to be found.

As conduction in the liquid in the vertical direction is being neglected (see Appendix), the infinite solution for the temperature profile in the liquid is valid for all positive values of z

$$T = \frac{q}{4\pi\lambda}\ln(4\kappa t/a^2C). \quad (13)$$

It follows that the heat sink q, which represents the heat loss of the wire to the surrounding liquid, can be written as:

$$q = \frac{4\pi\lambda}{\ln(4\kappa t/a^2C)}T. \quad (14)$$

The heat loss coefficient α follows directly from this. A minimum value (the worst case) for this α is found if one uses for t the time after which the experiment is stopped.

$$\alpha = \frac{4\pi\lambda}{\ln(4\kappa t_e/a^2C)}. \quad (15)$$

Substitution of this value into equation (12) finally gives the upper limit for the error due to the cooling down of the two ends of the wire

$$E = \frac{a}{L}\sqrt{(\lambda_1/\lambda)\ln(4\kappa t_e/a^2C)} \times\left(\frac{\operatorname{erf}\sqrt{\beta t} - 2\sqrt{\beta t/\pi}\,e^{-\beta t}}{1-e^{-\beta t}}\right). \quad (16)$$

In most applications $\beta t \gg 1$, in which case the term between brackets in equation (16) has a value of 1; so the final result is

$$E = \frac{a}{L}\sqrt{(\lambda_1/\lambda)\ln(4\kappa t_e/a^2C)}. \quad (17)$$

A measurement of the thermal conductivity of toluene at 25°C with a platinum wire 20 cm long and having a radius of 10 μm, using a measurement time of 1 s, will be subject to an error of 0.33% if the heat input is 0.35 W m^{-1}.

The implication of this small value for the end-effect is that an attempt at practical elimination by means of potential leads or compensating leads is, because of complexities, very likely to introduce errors larger than that which should be eliminated.

The final result is exactly the same as was found by Haarman [19] but the present solution is based on a time-dependent temperature profile in the liquid. The upper error for the end-effect is therefore determined with a high degree of reliability which Haarman's method cannot guarantee.

The method presented applies only when the convective effects are smaller than the axial conduction effects or of the same order.

For the analysis in the next section it is convenient to define the effective length of the conductive end-effect (l_c).

The errors due to the conductive end-effect can also be defined as

$$E = 2l_c/L \quad (18)$$

and its corresponding effective length is thus

$$l_c = \frac{a}{2}\sqrt{(\lambda_1/\lambda)\ln(4\kappa t_e/a^2C}. \quad (19)$$

4. ERROR DUE TO THE CONVECTIVE END-EFFECT

In this section the influence of convection on the end-effect is estimated. An accurate analysis would not be justified, due to the great sensitivity to verticality, but cannot be done in any case because of different experimental boundary conditions. The error calculated in this section is likely to be an overestimation (worst case), but is very useful for design purposes (instrument as well as measuring procedure).

While the wire is being heated, a convectional liquid flow around the wire will arise. This flow shifts the whole thermal boundary layer upwards, thus causing the bottom end-effect to lengthen and the top end-effect to shorten. At a certain time, the top end-effect will have vanished almost completely, whereas at the bottom it will still be growing. To determine the time at which this will occur, the penetration length of the convectional flow must be determined as a function of time.

After this time, the analysis given in Section 3 is no longer valid (and neither are those of Blackwell [16, 17], Haarman [19] or Healy *et al.* [18]) and another estimate of the end-effect must be made.

Therefore the behaviour of the convectional flow around the wire must be studied first.

4.1. *Velocity and penetration profile around the wire*

Goldstein and Briggs [21] presented the solution of the free convection around a heated wire as a complicated infinite integral. The derivation, which was not given, was also made by the author to conclude that the integral, as stated, was correct. Their statement of the problem and their solution are briefly given in this subsection. It is important to note that the temperature and velocity profiles 'far' away from the ends of the wire are considered; the dependence on the axial variable is then negligible. Three coupled differential equations for conservation of mass, momentum and energy must be solved.

The Boussinesq approximation states that the temperature dependence of density can only be used to describe the buoyancy force, but that for all other purposes all physical properties are constant. This approximation is used to simplify the equations. From the mass conservation equation, which reduces to $\nabla \cdot v = 0$, and the assumption that far from the end of the wire all derivatives with respect to the axial variable are zero, it follows that the convective operator $(v \cdot \nabla)$, which occurs in both the energy and momentum equations, yields a null operator. This uncouples the energy and momentum equations; the solution of the former, though, appears in the source term of the latter. Neglecting viscous dissipation terms, the resulting equations with the appropriate boundary conditions are [21]:

$$\frac{\partial T}{\partial t} = \frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{20}$$

$$\frac{\partial u}{\partial t} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + g\beta T \tag{21}$$

$$T = 0 \quad \text{at} \quad t \leqslant 0,\ r \geqslant a \tag{22}$$

$$T = 0 \quad \text{at} \quad r \to \infty,\ t > 0 \tag{23}$$

$$\frac{\partial T}{\partial r} = -\frac{Q}{2\pi a\lambda} \quad \text{at} \quad r = a,\ t > 0 \tag{24}$$

$$u = 0 \quad \text{at} \quad t \leqslant 0,\ r > a \tag{25}$$

$$u = 0 \quad \text{at} \quad r \to \infty,\ t > 0 \tag{26}$$

$$u = 0 \quad \text{at} \quad r = a,\ t > 0. \tag{27}$$

The solution of this set of equations in the Laplace domain can be found using standard methods. The solution for the velocity in the axial direction is (not mentioned in [21]):

$$\bar{v}(r,p) = \frac{g\beta Q Pr}{2\pi a\lambda\nu(Pr-1)}\frac{1}{pq^3} \times \left[\frac{K_0(qa)}{K_1(qa)}\frac{K_0(qr/\sqrt{Pr})}{K_0(qa/\sqrt{Pr})} - \frac{K_0(qr)}{K_1(qa)}\right]. \tag{28}$$

The penetration length, which follows from integrating the velocity from time zero to an arbitrary point in time, can now be simply found by dividing the equation for the velocity by the Laplace variable. The resulting equation for the penetration length is thus:

$$\bar{s}(r,p) = \frac{g\beta Q Pr}{2\pi a\lambda\nu(Pr-1)}\frac{1}{p^2q^3} \times \left[\frac{K_0(qa)}{K_1(qa)}\frac{K_0(qr/\sqrt{Pr})}{K_0(qa/\sqrt{Pr})} - \frac{K_0(qr)}{K_1(qa)}\right]. \tag{29}$$

Application of the usual Bromwich contour integral for the inverse Laplace transform when the function has a branch point at $p = 0$, yields for velocity and penetration length in the real time domain [21]:

$$u(r,t) = \frac{g\beta Qt}{\pi^2\lambda(Pr-1)\tau}\int_0^\infty \frac{(1 - V^2\tau + e^{-V^2\tau})}{V^4}\xi(R,V,Pr)\,dV \tag{30}$$

$$s(r,t) = \frac{g\beta Qt^2}{\pi^2\lambda(Pr-1)\tau^2}\int_0^\infty \frac{(-1 + V^2\tau - V^4\tau^2 + e^{-V^2\tau})}{V^6}\xi(R,V,Pr)\,dV \tag{31}$$

$$\begin{aligned}\xi(R,V,Pr) = {} & \frac{J_0(RV)Y_1(V) - J_1(V)Y_0(RV)}{J_1^2(V) + Y_1^2(V)} + \frac{2}{\pi V}\frac{J_0(RV/\sqrt{Pr})J_0(V/\sqrt{Pr}) + Y_0(RV/\sqrt{Pr})Y_0(V/\sqrt{Pr})}{[J_0^2(V/\sqrt{Pr}) + Y_0^2(V/\sqrt{Pr})][J_1^2(V) + Y_1^2(V)]} \\ & - \frac{[J_0(V)J_1(V) + Y_1(V)Y_0(V)][J_0(RV/\sqrt{Pr})Y_0(V/\sqrt{Pr}) - Y_0(RV/\sqrt{Pr})J_0(V/\sqrt{Pr})]}{[J_0^2(V/\sqrt{Pr}) + Y_0^2(V/\sqrt{Pr})][J_1^2(V) + Y_1^2(V)]}.\end{aligned} \tag{32}$$

The integrals were calculated numerically and the results are presented similarly to those of Goldstein and Briggs [21] for their case $Pr = 1$ (see Figs. 2a and 2b).

For our case it is sufficiently accurate to approximate the maximum penetration length using the following formula, found by a least-squares analysis of the results presented in Fig. 2a:

$$l_p = 0.0055\frac{g\beta Qt^2}{\lambda}Pr^{-0.62}(\log_{10}\tau)^{0.84}. \tag{33}$$

The error made by applying this formula is typically 20%. The distance from the wire at which this maximum penetration occurs is determined roughly by:

$$\frac{(R-1)}{\sqrt{\tau}} = 0.2 \quad \text{or at} \quad r = a + 0.2\sqrt{\kappa t}. \tag{34}$$

4.2. *The influence of the penetration length on the end-effect*

For the purpose of a worst-case approximation, it can be assumed that over the penetration length the wire has been cooled down to the environmental

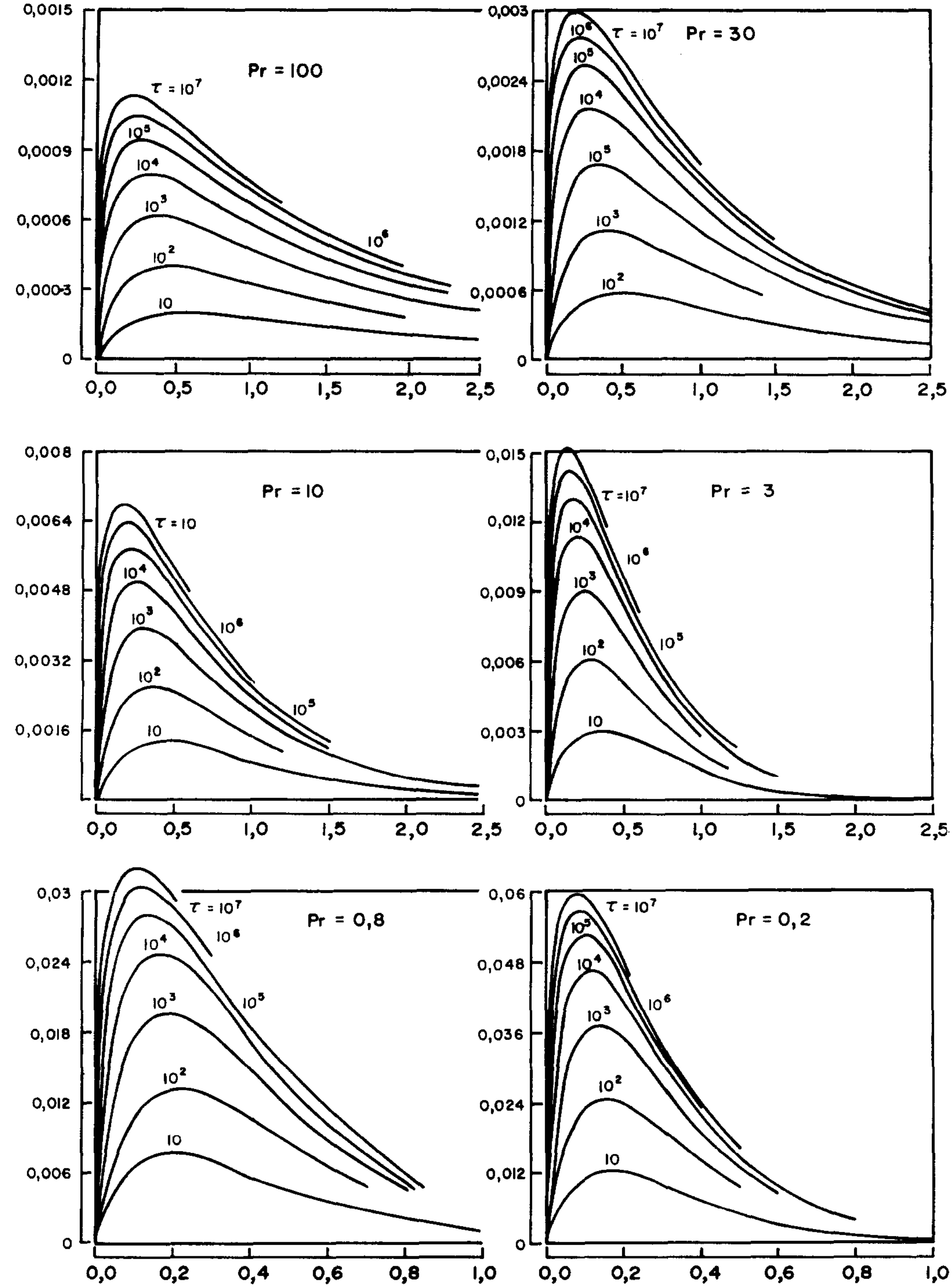

FIG. 2a. Penetration profiles for various Pr with τ as a parameter. The vertical axis shows values of $[s(r,t)\lambda]/g\beta Qt^2$ and the horizontal axis of $(R-1)/2\sqrt{\tau}$.

temperature by the inflow of liquid at this temperature (Fig. 3). It is also assumed that the conductive end-effect and the convective end-effect do not interfere and that the two effects can be superimposed.

Due to the convective motion in the liquid, the conductive end-effects are shifted upwards causing the top end-effect to diminish (by the continuous inflow of hot liquid from below) and the bottom end-effect to elongate (Fig. 3b). The convectional current will now start to heat up the top support as well. Eventually the top end-effect will be small at the transition time (Fig. 3c). This 'transition time' is defined as the time at which

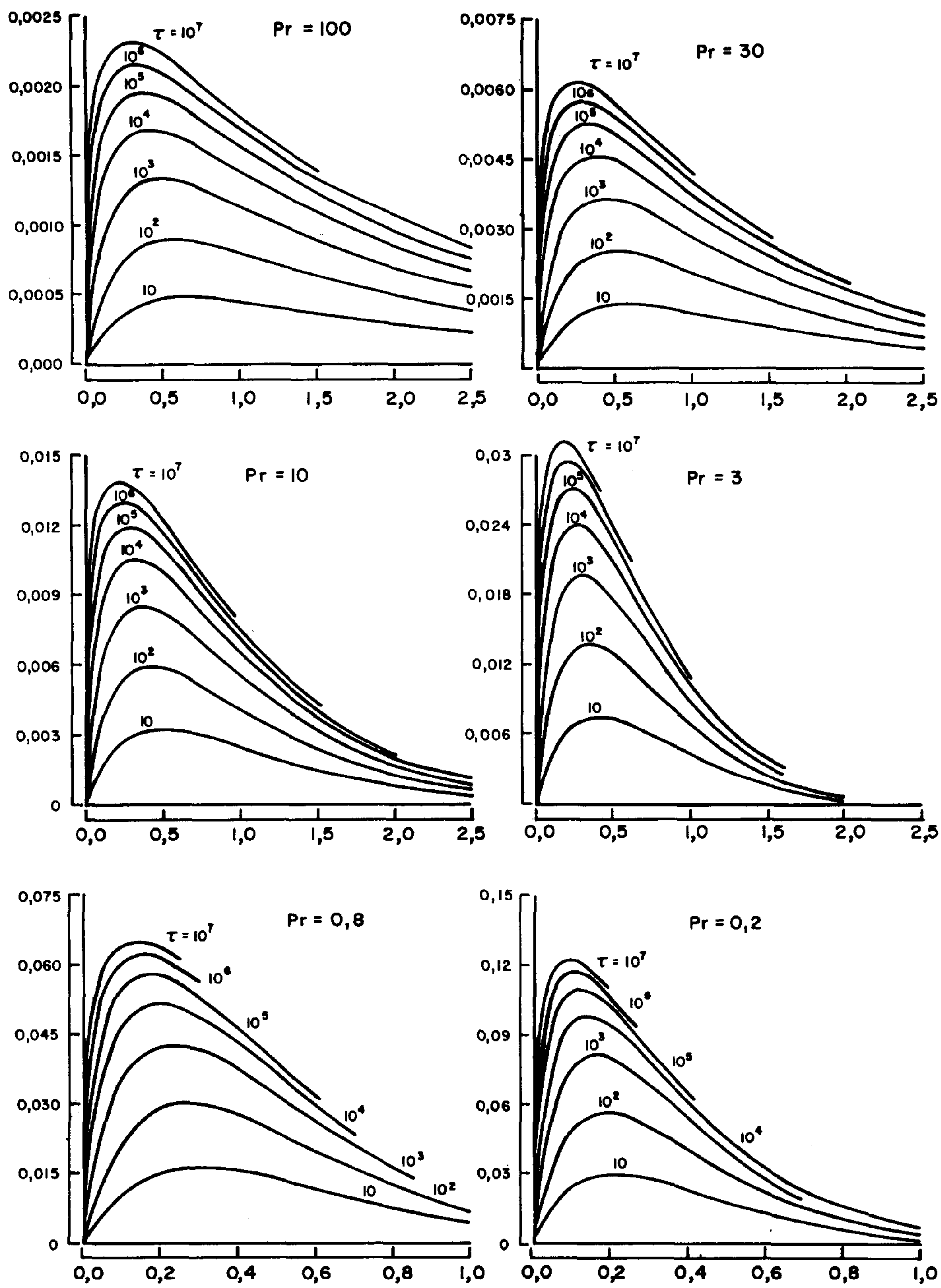

FIG. 2b. Velocity profiles for various Pr with τ as a parameter. The vertical axis shows values of $[u(r, t)\lambda]/g\beta Qt$ and the horizontal axis of $(R-1)/2\sqrt{\tau}$.

the conductive penetration length and the convective penetration length are equal. The convective and conductive end-effect will then be approximately equal in size. After the transition time (Fig. 3d), the total end-effect can be estimated by summing half the conductive end-effect and the whole convective end-effect. The analysis made in Section 3 is no longer sufficient, and the error is found from:

$$E = \frac{l_c + l_p}{L}. \tag{35}$$

The transition time after which the convective end-

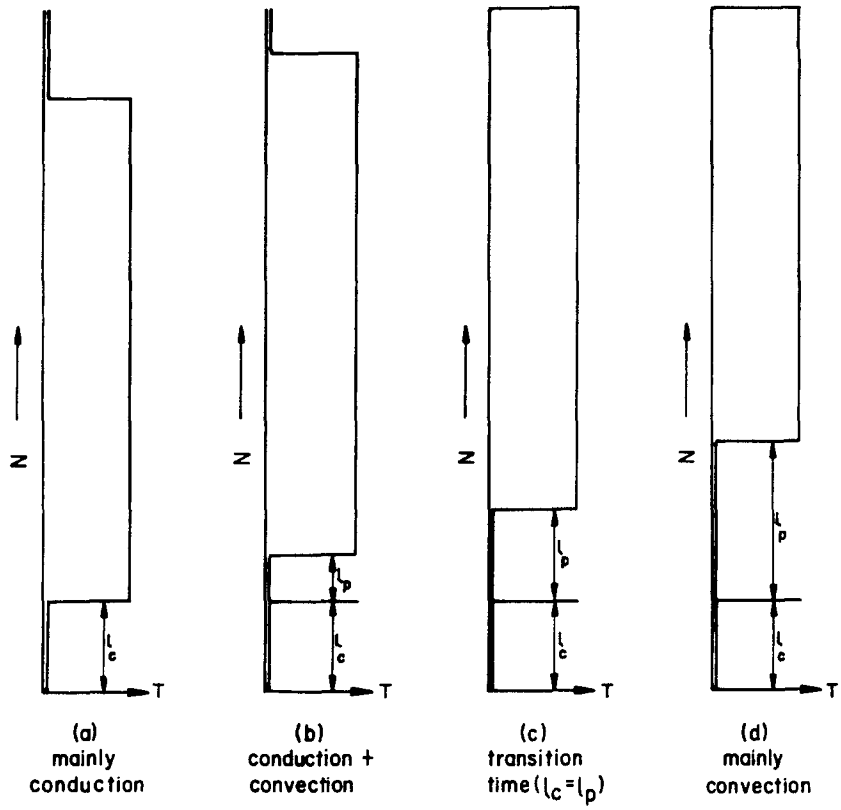

FIG. 3. Assumed temperature profiles at different stages of the measurement.

effect dominates over the conductive end-effect can now be determined by equating the effective length of the conductive end-effect [equation (19)] and the penetration length [equation (33)] and solving for t from this implicit equation:

$$t^2[\ln(\kappa t/a^2)]^{0.84} = \frac{183a\,Pr^{0.62}}{g\beta Q}\sqrt{\lambda\lambda_1}\ln(4\kappa t_e/a^2C). \quad (36)$$

The total error due to the end-effects after the transition time can thus be written as:

$$E = \frac{a}{2L}\sqrt{(\lambda_1/\lambda)\ln(4\kappa t_e/a^2C)} + 0.0055\frac{g\beta Qt^2}{\lambda L}Pr^{-0.62}[\log_{10}(\kappa t/a^2)]^{0.84}. \quad (37)$$

In reality the convective and conductive end-effects are not as strictly separated at this formula implies, the reason being that near the end of the wire convective terms should be included in the energy and momentum equation. The analysis in this section is therefore not reliable enough to warrant making a theoretical correction afterwards, but, as stated before, it can be used for design purposes.

Example. A measurement on toluene at 25°C with a wire 20 μm in diameter in which the heat input is 0.35 W m^{-1} yields a transition time of 1.5 s.

5. ONSET OF CONVECTION

The transient hot-wire method is preferred to the steady-state method because the effect of convection can be clearly seen in the measured transient as a deflection from a straight line in the T vs $\ln(t)$ plot, and so can be eliminated.

That convection plays an important role cannot be denied since several researchers [22–24] have found a strong correlation between the onset of the disturbance and a certain Rayleigh number. The value of this Rayleigh number varies among the different researchers and is therefore likely to be dependent on the experimental set-up.

The real cause of this disturbance is still not well understood. It is very unlikely, however, that the disturbance is caused by transition from a laminar to a turbulent flow profile. Several factors could cause this effect, and it is likely that some will occur simultaneously as well.

From the analysis in the previous section, it can be deduced that this effect is caused by an extended convective end-effect. In that case the length of the wire would be an important parameter.

The verticality of the wire is reported to have a considerable influence [5] on the onset time. This makes sense, since convectional currents are likely to transport 'cold' liquid in a vertical direction to a wire that is not mounted vertically, thus cooling it down. In the example given above, the penetration length is 0.33 mm at the transition time (1.5 s) which is well below the onset time of convection. At this stage, the penetration length is already more than 16 wire diameters. After 4 s this will have increased to c. 130 wire diameters and the sensitivity to verticality becomes obvious.

Another possible source of this disturbance was observed in an experiment using a very high heat input per unit length to increase the wire temperature ca. 30°C. During this experiment the wire made an unexpected sideways movement of at least 1 mm (50 wire diameters) even though the wire was under tension. This movement may be caused by convectional currents around local disturbances either in the liquid or on the wire, or it may be an electromagnetic effect (e.g. due to the terrestial field); it may also be caused by thermal expansion of the wire which is not immediately compensated for by the weight or spring tensioning the wire. If this phenomenon also occurs at lower heat inputs, which is very likely, the result could be a mixing of the thermal boundary layer around the wire which would cool down the wire.

With longer times this mixing would become more pronounced because of the fast-increasing velocities in the boundary layer. The consequence for thermal conductivity measurements will be that the measured temperature increase will deviate from the expected linear relationship.

An important consequence of the mixing effect is that the wire thickness becomes important: the thinner the wire, the more sensitive it will be to this mixing effect (lower inertia and more cooled down by the mixing due to a smaller heat capacity).

6. CONCLUSIONS

The conductive end-effect, which is dominant in the first part of the measurement, is almost constant during this period. The worst-case analysis presented will not differ too much from most practical cases so that its result can be used to correct the measured thermal conductivity, if necessary.

The convective end-effect, however, causes an error that increases roughly with the time squared. This error cannot be accurately quantified due to the movement of fluid and wire, so it is not accurate to apply the theoretically derived correction factor to the measured result. It is therefore advisable to do the measurement quickly.

Because the end-effect is normally negligible for small times, it is not necessary to introduce practical corrections when using a rapid measurement technique. The unavoidable addition to the total systematic error caused by such a correction may be of the same size or larger than the error which is to be eliminated. The size of this systematic error is clearly demonstrated by the differences of up to 2% in measurement results of various experimenters, many of whom claim an accuracy of 0.2%. This level of accuracy probably represents the reproducibility or precision of the measurement and not the absolute accuracy of the method.

7. REFERENCES

1. F. J. Dietz, J. J. de Groot and E. U. Franck, High pressure thermal conductivity measurements of water with ac and dc transient hot-wire methods, *Proc. 9th Int. Conference on the Properties of Steam*, pp. 425–430 (1979).
2. J. J. de Groot, J. Kestin and H. Sookiazian, Instrument to measure the thermal conductivity of gases, *Physica* **75**, 454–482 (1974).
3. E. N. Haran and W. A. Wakeham, A transient hot-wire cell for thermal conductivity measurements over a wide temperature range. *J. Phys. E: Scient. Instrum.* **15**, 839–842 (1982).
4. J. Kestin and W. A. Wakeham, A contribution to the theory of the transient hot-wire technique for thermal conductivity measurements, *Physica* **92A**, 102–116 (1978).
5. C. A. Nieto de Castro, J. C. G. Calado, W. A. Wakeham and M. Dix, An apparatus to measure the thermal conductivity of liquids, *J. Phys. E: Scient. Instrum.* **9**, 1073–1080 (1976).
6. J. D. Raal and R. L. Rijsdik, Measurement of alcohol thermal conductivities using a relative strain-compensated hot-wire method, *J. chem. Engng Data* **26**, 351–359 (1981).
7. H. M. Roder, A transient hot-wire thermal conductivity apparatus for fluids, *J. Res. natn. Bur. Stand.* **86**, 457–493 (1981).
8. Y. Nagaska and A. Nagashima, Absolute measurement of thermal conductivity of electrically conducting liquids by the transient hot-wire method, *J. Phys. E: Scient. Instrum.* **14**, 1435–1440 (1981).
9. P. Andersson and G. Backstrom, Thermal conductivity of solids under pressure by the transient hot-wire method, *Rev. scient. Instrum.* **47**, 205–209 (1976).
10. D. G. Gillam, L. Romben, H. Nissen and O. Lamm, Accurate determination of thermal conductivities, *Acta chem. scand.* **9**, 641–656 (1955).
11. E. McLaughlin, J. F. T. Pittman, Determination of the thermal conductivity of toluene, a proposed data standard, from 180 to 400 K under saturation pressure by the transient hot-wire method. II. New measurements and a discussion of existing data, *Phil. Trans. R. Soc.* **A270**, 579–602 (1971).
12. Y. Nagasaka and A. Nagashima, Precise measurements of the thermal conductivity of toluene and *n*-heptane by the absolute transient hot-wire method, *Ind. Engng Chem. Fundam.* **20**, 216–220 (1981).
13. S. Takizawa, H. Murata and A. Nagashima, Measurement of the thermal conductivity of liquids by transient hot-wire method. I: Measurement at atmospheric pressure, *Bull. JSME* **21**, 273–278 (1978).
14. W. N. Trump, H. W. Luebke, L. Fowler and E. M. Emery, Rapid measurement of liquid thermal conductivity by the transient hot-wire method, *Rev. scient. Instrum.* **48**, 47–51 (1977).
15. P. Grassmann and W. Straumann, Ein instationares Verfahren zur Messung der Wärmeleitfähigkeit von Flüssigkeiten und Gasen, *Int. J. Heat Mass Transfer* **1**, 50–54 (1960).
16. J. H. Blackwell, Radial–axial heat flow in regions bounded internally by circular cylinders, *Can. J. Phys.* **31**, 472–479 (1953).
17. J. H. Blackwell, The axial flow error in the thermal conductivity probe, *Can. J. Phys.* **34**, 412–417 (1956).
18. J. J. Healy, J. J. de Groot and J. Kestin, The theory of the transient hot-wire method for measuring thermal conductivity, *Physica* **82C**, 392–408 (1976).
19. J. W. Haarman, A contribution to the theory of the transient hot-wire method, *Physica* **52**, 605–619 (1971).

20. J. K. Horrocks and E. McLaughlin, Unsteady state measurements of the thermal conductivities of liquid polyphenyls, *Proc. R. Soc.* **A273**, 259–274 (1963).
21. R. J. Goldstein and D. G. Briggs, Transient free convection about vertical plates and circular cylinders, *Trans. Am. Soc. mech. Engrs*, Series C, *J. Heat Transfer* **86**, 490–500 (1964).
22. P. Grasmann, W. Straumann, F. Widmer and W. Jobst, Measurement of thermal conductivities by an unsteady state method, *Proc. Int. Res. Thermodynamic Properties, Paper 2nd Symposium on Thermophysical Properties*, pp. 447–452 (1962).
23. E. F. M. van der Held and F. G. van Drunen, A method of measuring the thermal conductivities of liquids, *Physica* **15**, 865–881 (1949).
24. H. Ziebland (collator) and K. N. Marsh (editor), Recommended reference materials for realization of physicochemical properties, *Pure Appl. Chem.* **53**, 1863–1877 (1981).

APPENDIX

Heuristic analysis of the ratio of the heat flows in the liquid and the wire

In the analysis given in Section 2, it is assumed that the temperature rise $T(r, z, t)$ in the liquid and in the wire is zero at $z = 0$. From this it seems quite reasonable to assume that the solution for the temperature distribution in the liquid near the end of the wire can be approximated by the temperature distribution far from the wire multiplied by a z-dependent function of an arbitrary form

$$T = \frac{q}{4\pi\lambda} E_1\left(\frac{r^2}{4\kappa t}\right) f(z), \quad 4\kappa t/r^2 \ll 1. \tag{A1}$$

The heat flux in axial direction in the liquid is then given by

$$\phi_1'' = -\lambda \frac{\partial T}{\partial z} = -\frac{q}{4\pi} E_1\left(\frac{r^2}{4\kappa t}\right) f'(z) \tag{A2}$$

and the total amount of heat transported through the liquid to the supports is then

$$\int_{r=a}^{\infty} \phi_1'' 2\pi r \, dr = -q\kappa t \int_{\xi_a}^{\infty} E_1(\xi) \, d\xi \, f(\kappa) \tag{A3}$$

where

$$\xi = \frac{r^2}{4\kappa t} \quad \text{and} \quad \xi_a = \frac{a^2}{4\kappa t}. \tag{A4}$$

The total heat flow in the wire is easier to calculate, since the temperature over its cross-section is constant and equal to the temperature of the liquid at a distance corresponding to the radius of the wire.

$$\phi_\omega = -\lambda_1 \pi a^2 \left.\frac{\partial T}{\partial r}\right|_{r=0} = -\frac{\lambda_1}{\lambda} \frac{qa^2}{4} E_1\left(\frac{a^2}{4\kappa t}\right). \tag{A5}$$

The ratio of both heat flows is then

$$\frac{\phi_\omega}{\phi_1} = \frac{\lambda_1}{\lambda} \frac{\xi_a E_1(\xi_a)}{\int_{\xi_a}^{\infty} E_1(\xi) \, d\xi}. \tag{A6}$$

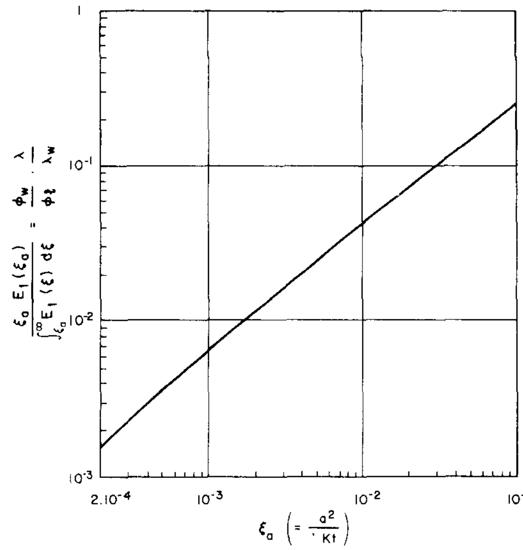

FIG. A1. Ratio of the heat flow through the wire and through the liquid as a function of the measurement time, the wire radius and material properties.

In Fig. A1 the last term is plotted. It is easy to see that for a measurement in toluene with a 20-μm-thick wire, the heat flows are approximately the same when the transition time is reached. In this case the conductive approach to the end-effect near the transition time is inaccurate. As mentioned earlier, this approach would be inaccurate anyway, since the convective currents will have distorted the heat flux in the liquid completely. If shorter measuring times are employed or if thicker wires are used, the heat flow ratio becomes more favourable, which increases the level of confidence of the analysis in Section 4.

L'ERREUR PAR EFFECT D'EXTREMITE DANS LA DETERMINATION DE LA CONDUCTIVITE THERMIQUE A PARTIR D'UN APPAREIL A FIL CHAUD

Résumé—La plupart des instruments de mesure de conductivité thermique basés sur la technique de la source rectiligne fournit une manière pratique de compensation de l'effet de bout. Une analyse théorique montre que l'erreur peut être rendue très petite, si bien qu'il n'est pas toujours nécessaire de procéder à une correction de façon pratique. On donne une limite supérieure pour cette erreur. Dans cette analyse, on fait une distinction entre les effets de bout conductif et convectif; dans le première partie de la mesure l'effet conductif domine, mais son influence est rapidement dépassée par celle de l'effet convectif.

DER FEHLER DURCH END-EFFEKTE BEI DER BESTIMMUNG DER WÄRMELEITFÄHIGKEIT MIT EINER HITZDRAHTAPPARATUR

Zusammenfassung—Bei den meisten Geräten zur Bestimmung der Wärmeleitfähigkeit, die auf der instationären Hitzdraht-Technik basieren, wird der auftretende End-Effekt praktisch berücksichtigt. Eine theoretische Untersuchung zeigt, daß der durch den End-Effekt verursachte Fehler sehr klein gehalten werden kann, so daß es im praktischen Fall nicht immer erforderlich ist, eine Korrektur vorzunehmen. Eine obere Grenze für den Fehler wird hergeleitet. In der Untersuchung wird unterschieden zwischen dem konduktiven und dem konvektiven End-Effekt. Im ersten Teil einer Messung überwiegt der konduktive Effekt, aber sein Einfluß wird schnell durch den konvektiven Effekt überdeckt.

ПОГРЕШНОСТЬ ИЗ-ЗА КОНЦЕВЫХ ПОТЕРЬ ПРИ ОПРЕДЕЛЕНИИ ТЕПЛОПРОВОДНОСТИ НА УСТАНОВКЕ С НАГРЕТОЙ НИТЬЮ

Аннотация—В большинстве установок для измерения теплопроводности, использующих нестационарный линейный источник, предусмотрена практическая компенсация концевых эффектов. Теоретический анализ показывает, что погрешность, вызванная этим эффектом, может быть очень мала, так что не всегда ее нужно корректировать в практических целях. Установлен верхний предел погрешности. При анализе отдельно исследуются концевые потери из-за конвекции и теплопроводности: на первой стадии измерения преобладает кондуктивный эффект, в дальнейшем подавляющим становится конвективный.